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Abstract

A recent renewed claim that the instanton method and the WKB method do not lead to identical results in the 1–loop approximation is repudiated. It is pointed out that the WKB method, or its equivalent, yields the instanton result provided the boundary conditions are imposed on wave functions which are properly matched in domains of overlap.

1 Introduction

We comment on ref.[1]. Ever since the instanton method of evaluating path integrals became popular, the question was raised as to whether this method and that of the Schrödinger equation lead to identical results in the dominant approximation. In view of the significance of both methods this is an important question. Generally one would argue that in the 1–loop approximation of the path integral the results of both methods should agree – and, in fact, this is the widely accepted opinion. However, there are few models which allow explicit verification, and some detailed investigations of even these are not well known. Thus recently in a study of Lamé instantons [1], a parallel consideration of the WKB method was found to lead to a result

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which is off by an overall factor of $\sqrt{e/\pi}$ – very similar to an observation made long ago in ref.[2] in the case of the well known double well potential. In the following we point out that in both cases, the WKB approximations employed are too simple, and complete agreement between both methods in all the examples referred to can be achieved with more detailed investigation [3]. The best known examples for such considerations are the scalar theory with the double well potential and the sine–Gordon theory of the periodic potential. The case considered in ref.[1] with the Lamé potential is not so well known but has recently found widespread consideration in various contexts, such as supersymmetric quantum mechanics [4] and spin tunneling (cf. e.g. [5]), and includes in a particular limit the sine–Gordon case. In the following we recall the level splitting calculated long ago for the Lamé (and more generally ellipsoidal) wave equation [6] and consider limits and special cases to demonstrate the agreement with the result of the instanton method, in particular that of ref.[1]. For details of the calculations we refer to the literature, where many details are given.

2 The Lamé level splitting and consequences

The Lamé equation [7]

$$y'' + \left\{\Lambda - \kappa^2 sn^2 u\right\} y = 0, \quad \kappa^2 = n(n+1)k^2, \tag{1}$$

with elliptic modulus $|k| < 1$ and $n > -1/2, 0 < u < 2\mathcal{K}$, can be looked at as a Schrödinger equation with periodic potential $\kappa^2 sn^2 u$, where snu is one of the Jacobian elliptic functions with period $2\mathcal{K}$, and $\mathcal{K}$ is one of the complete elliptic integrals. (In the comparison with the Schrödinger equation, the usual factor $-\hbar^2/2m$ in front of the second derivative has to be kept in mind, where m is the mass). The perturbatively derived wave functions of ref.[6] (for large values of κ^2), when matched in domains of overlap and so extended over the entire domain of the variable u, are subject to periodic boundary conditions which define two pairs of eigenfunctions, in each case with one even and one odd, of periods $2\mathcal{K}$ and $4\mathcal{K}$ respectively. These four conditions together imply for large values of κ^2 the following asymptotic expansion of the eigenvalues

Λ as shown in ref.[6]:

$$\Lambda_{\pm}(q_0) = \Lambda(q_0) \pm \frac{2\kappa\left(\frac{2}{\pi}\right)^{\frac{1}{2}}}{[\frac{1}{2}(q_0-1)]!}\left(\frac{1+k}{1-k}\right)^{-\frac{\kappa}{k}}\left(\frac{8\kappa}{1-k^2}\right)^{\frac{1}{2}q_0}\left[1+O\left(\frac{1}{\kappa}\right)\right] \qquad (2)$$

Here $q_0 = 2N+1, N = 0, 1, 2, \cdots$ and $\Lambda(q_0)$ is the purely perturbative contribution which represents effectively the eigenvalues of degenerate oscillators of the periodic potential in the case of very high barriers. It is the boundary conditions imposed on the perturbatively derived solutions which yield the nonperturbative effects equivalent to those of the instanton. Thus the factor

$$\left(\frac{1+k}{1-k}\right)^{-\kappa/k}$$

is, in fact $exp(-S_0)$, where S_0 is the Euclidean action of the instanton.

We first verify the result (2) by reduction to the sine–Gordon case. With $\kappa = \pm 2h$ finite while $n \rightarrow \infty$ and $k \rightarrow 0$ the Jacobian elliptic function snu reduces to $\sin u$ and eq.(1) becomes by replacing u by $x \pm \pi/2$ the Mathieu equation

$$y'' + \left\{\lambda - 2h^2\cos^2 2x\right\}y = 0, \quad \lambda \equiv \Lambda - 2h^2. \qquad (3)$$

Under the conditions stated the eigenvalues become

$$\lambda_{\pm}(q_0) = \lambda(q_0) \pm \frac{4h(\frac{2}{\pi})^{\frac{1}{2}}(16h)^{\frac{q_0}{2}}}{[\frac{1}{2}(q_0-1)]!}e^{-4h}\left[1+O\left(\frac{1}{h}\right)\right] \qquad (4)$$

in agreement with established results in this case [8, 9, 10].

Next we consider k approaching 1 in the case of the Lamé eigenvalues (2) (terms up to and including those of $O(1/\kappa^2)$ in the level splitting and up to and including those of $O(1/\kappa^4)$ in the perturbative part have been given in ref.[6] for any q_0). One readily obtains

$$\Lambda_{\pm}(q_0) = \Lambda(q_0) \pm \frac{(8\kappa)^{\frac{q_0}{2}+1}(1-k)^{\kappa-\frac{1}{2}q_0}}{[\frac{1}{2}(q_0-1)]!(2\pi)^{1/2}2^{\kappa+1+\frac{q_0}{2}}}\left[1+(1-k)\{\kappa(\frac{1}{2}-\ln 2)+\frac{q_0}{4}\}+O\left(\frac{1}{\kappa}\right)\right] \qquad (5)$$

For the two lowest levels $q_0 = 1$ and one obtains

$$\Lambda_{\pm}(1) = \Lambda(1) \pm \frac{(4\kappa)^{3/2}(1-k)^{\kappa-\frac{1}{2}}}{(2\pi)^{1/2}2^{\kappa}}\Big[1+(1-k)\{\kappa(\frac{1}{2}-\ln 2)+\frac{1}{4}\}+O\Big(\frac{1}{\kappa}\Big)\Big] \quad (6)$$

Thus the separation of the two lowest levels is

$$\triangle\Lambda(1) \simeq \frac{2(4\kappa)^{3/2}(1-k)^{\kappa-\frac{1}{2}}}{(2\pi)^{1/2}2^{\kappa}}\Big[1+(1-k)\{\kappa(\frac{1}{2}-\ln 2)+\frac{1}{4}\}+O\Big(\frac{1}{\kappa}\Big)\Big] \quad (7)$$

This result agrees with formula (13) of ref.[1] in the limit of (in our notation) large κ and $k \to 1$ (in particular this agrees also with our power $\kappa - 1/2$ of $(1-k)$ where in [1] the $-1/2$ has been ignored). The higher order contributions are, presumably, somewhat model or approximation dependent. We thus have agreement with the instanton result of ref.[1] (there ν is our k^2, so that for $k \to 1$ one has $(1-\nu) \to 2(1-k)$).

Hence in the case of the periodic cosine potential, as well as in the case of the Lamé potential, the WKB method – or its equivalent of refs.[6, 8] – yields the same result as the instanton method in the 1–loop approximation, as one would expect. In ref.[1] reference is made to the well known case of the double well potential. For this case also it has been shown in ref. [10] that the method of matched perturbation expansions – which one might argue amounts effectively to the same as the WKB method (rather than a WKB approximation) – yields the same result as the instanton method, again for any arbitrary level, whether ground state or excited [10].

3 Conclusions

In the above we have demonstrated that the WKB method or one of its equivalents like the method of matched asymptotic expansions of refs.[6, 8], which, incidentally, was also developed and used to determine the large order behaviour of the perturbation expansion (cf. ref.[12]), leads to the same result as the instanton calculation in the 1–loop approximation. One can see, however, that a more detailed consideration is required than a simple WKB approximation, and in fact, we may conclude that to obtain from the Schrödinger equation and perturbation theory results agreeing with those of the path integral method with expansion around the instanton, one has to use the full method of matched asymptotic expansions as developed in refs.[8] and [6] (which has also been applied to other cases such as spheroidal wave equations[13]). In purely quantum mechanical cases, such as those considered above, the method of matched asymptotic expansions seems to be simpler and yields the splitting of excited oscillator states with the same ease as that of the ground state, whereas in the case of the path integral method, one has to use periodic instantons, as in e.g. refs. [11, 14]. This may be worth noting in connection with models of spin tunneling which attracted considerable interest recently, since these can – in certain cases and with certain approximations (and coherent states) – be related to periodic differential equations. Thus in ref.[5] the case of the Hamiltonian $\hat{H} = K_1\hat{S}_z^2 - K_2\hat{S}_x^2$ describing a ferromagnetic particle with large spin has been considered and related to Mathieu and Lam*é* equations. In particular the specific Lam*é* instanton of ref. [1] has been used in ref.[15].

Acknowledgement: J.–z. Z. acknowledges support of the Deutsche Forschungsgemeinschaft and Y. Z. that of an A. v. Humboldt–Foundation Fellowship.

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